

Dynamics of Particles Around a Schwarzschild-like Black Hole in the Presence of Quintessence and Magnetic Field

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Abstract: We investigate the dynamics of a neutral and a charged particle around a static and spherically symmetric black hole in the presence of quintessence matter and external magnetic field. We explore the conditions under which the particle moving around the black hole could escape to infinity after colliding with another particle. The innermost stable circular orbit (ISCO) for the particles are studied in detail. Mainly the dependence of ISCO on dark energy and on the presence of external magnetic field in the vicinity of black hole is discussed. By using the Lyapunov exponent, we compare the stabilities of the orbits of the particles in the presence and absence of dark energy and magnetic field. The expressions for the center of mass energies of the colliding particles near the horizon of the black hole are derived. The effective force on the particles due to dark energy and magnetic field in the vicinity of black hole is also discussed.

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I. INTRODUCTION

The accelerating expansion of the universe indicates the presence of elusive dark energy. The presence of dark energy is supported by several astrophysical observations including the study of Ia Supernova [1], cosmic microwave background (CMB) [2] and large scale structure (LSS) [3, 4]. The nature of dark energy is not understood until now. It is explained by cosmological models in which dominant factor of dark energy density may possess negative pressure such as cosmological constant Λ with a state parameter $w_q = -1$. There are other scalar field models that are proposed such as quintessence [5], phantom dark energy [6], k-essence [7], holographic dark energy[8] to name a few. Dark energy is approximately 70 percent of the energy density of the universe. If dark energy is dynamical, then naturally it will become more dominant in the future and will play a crucial role at all length scales. In this context, we study the motion of the particles around the black hole surrounded by dark energy and magnetic field.

The magnetic coupling process is responsible for attraction of black hole with its accretion disc [9]. According to this process, the angular momentum and energy are transferred from a black hole to its surrounding disc. There exists a sufficient strong magnetic field in the vicinity of black hole (Penrose effect for magnetic field) [10, 11]. Observational evidences indicate that magnetic field should be present in the vicinity of black holes [12]. This magnetic field arises due to plasma in the surrounding of black hole. The relativistic motion of particles in the conducting matter in the accretion disk may generate the magnetic field inside the disk. This field does not affect the geometry of the black hole, yet it affects the motion of charged particles and support them to escape [13, 14].

Bañados, Silk and West (BSW) proposed that some black holes may act as particle accelerators [15]. In the vicinity of extremal Kerr black hole, they have found that infinite center-of-mass energy (CME) can be achieved during the collision of particles. The BSW effect has been studied for different black hole spacetimes [15]-[25]. In this paper we obtain the CME expression for the colliding particles near horizon of Schwarzschild-like black hole surrounded by quintessence matter and BSW effect is studied for both neutral and charged particles.

Quintessence is defined as a scalar field coupled to gravity with the potential which decreases as field increases [26]. The solution for a spherically symmetric black hole surrounded

by quintessence matter was derived by Kiselev [27]. It has the state parameter in the range , $-1 < w_q < \frac{-1}{3}$. In this work we will focus on Kiselev solution. Null geodesics around Kiselev black hole have been studied in [28]. We consider the Kiselev solution in the presence of external axi-symmetric magnetic field, which is homogeneous at infinity. This magnetic field and quintessence matter strongly affect the dynamics of the particles and location of their innermost stable circular orbits (ISCO) around black hole. Before dealing with dynamics of a charged particle around black hole, we do analysis of a neutral particle without considering magnetic field. We construct the dynamical equations from the Lagrangian formalism which are not solvable via analytic methods. Even in the absence of black hole, charged particle motion in the non-uniform magnetic field which is chaotic [29, 30].

Main objective of our work is to study the circumstances under which the initially moving particle would escape from ISCO or its motion would remain bounded, after collision with some other particle. The effect of collision on the energy of the particle is also discussed. We have calculated the velocity of a particle required to escape to infinity and investigate some characteristics of the particle's motion moving around black hole. With the help of Lyapunov exponent, a comparison of the stability of orbits for massive and massless particles is also established [31].

The outline of the paper is as follows: We explain our model in section II and derive an expression for escape velocity of the neutral particle. In section III dynamics of a charged particle is discussed, we derive the equations of motion. In section IV dimensionless form of the equations are given. In section V CME expressions are derived for colliding particles. In section VI the Lyapunov exponent is calculated. In section VII the force on the charged particle is calculated. We discuss the trajectories for escape energy and escape velocity of the particle in section VIII. Concluding remarks are given in section IX. We use $(+, -, -, -)$ sign convention and gravitational units, $c = 1$, in this work.

II. DYNAMICS OF A NEUTRAL PARTICLE

The geometry of static spherically symmetric black hole surrounded by the quintessence matter (Kiselev solution) is given by [27]

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,$$

$$f(r) = 1 - \frac{2M}{r} - \frac{c}{r^{3w_q+1}}. \quad (1)$$

Here M is the mass of black hole, c is the quintessence parameter and w_q has range $-1 < w_q < \frac{-1}{3}$ while we will focus on $w_q = \frac{-2}{3}$. Metric (1) has curvature singularity at $r = 0$. For $f(r) = 0$ we get two values of r :

$$r_+ = \frac{1 + \sqrt{1 - 8Mc}}{2c}, \quad r_- = \frac{1 - \sqrt{1 - 8Mc}}{2c}. \quad (2)$$

The region $r = r_-$ corresponds to black hole horizon while $r = r_+$ represents the cosmological horizon. Therefore, r_- and r_+ are the two coordinate singularities in the metric (1). If $8Mc = 1$, we get the degenerate solution for the spacetime at $r_{\pm} = \frac{1}{2c}$ and if $8Mc > 1$, horizons do not exist. For very small value of c , $r_+ \approx \frac{1}{c}$. Furthermore, we can say that the restriction on c , is $c \leq \frac{1}{8M}$.

We discuss the dynamics of a neutral particle in the Schwarzschild-like background defined by (1). There are three constants of motion corresponding (1) in which two of them arise as a result of two Killing vectors [32]

$$\xi_{(t)} = \xi_{(t)}^\mu = \partial_t, \quad \xi_{(\phi)} = \xi_{(\phi)}^\mu = \partial_\phi. \quad (3)$$

where $\xi_t^\mu = (1, 0, 0, 0)$ and $\xi_\phi^\mu = (0, 0, 0, 1)$, Eq. (3) implies that, black hole metric (1) is invariant under time translation and rotation around symmetry axis ($\theta = 0$). The corresponding conserved quantities (conjugate momenta) are the energy per unit mass \mathcal{E} and azimuthal angular momentum per unit mass, L_z , respectively given by

$$\mathcal{E} \equiv f(r)\dot{t}, \quad (4)$$

$$-L_z \equiv \dot{\phi}r^2\sin^2\theta. \quad (5)$$

Here over dot represents differentiation with respect to proper time τ . The third constant of motion is the total angular momentum of the particle i.e.

$$L^2 = (r^2\dot{\theta})^2 + \frac{L_z^2}{\sin^2\theta} = r^2v_\perp^2 + \frac{L_z^2}{\sin^2\theta}. \quad (6)$$

Here we denote $v_\perp \equiv -r\dot{\theta}_o$. By using the normalization condition of 4-velocity $u^\mu u_\mu = 1$ and constants of motion (4) and (5), we get the equation of motion of neutral particle

$$\dot{r}^2 = \mathcal{E}^2 - \left(1 + \frac{L_z^2}{r^2 \sin^2 \theta}\right) f(r). \quad (7)$$

At the turning points of the moving particles from the trajectories $\dot{r} = 0$, hence equation (7) gives

$$\mathcal{E}^2 = \left(1 + \frac{L_z^2}{r^2 \sin^2 \theta}\right) f(r) \equiv U_{\text{eff}}, \quad (8)$$

where U_{eff} is the effective potential.

Consider a particle in the circular orbit $r = r_o$, where r_o is the local minima of the effective potential. This orbit exists for $r_o \in (4M, \infty)$. Generally for non-degenerate case ($r_+ \neq r_-$) the energy and azimuthal angular momentum corresponding to local minima r_o are

$$L_{zo} = \frac{\sqrt{cr_o^2 - 2M}}{\sqrt{c + \frac{6M - 2r_o}{r_o^2}}}, \quad (9)$$

and

$$\mathcal{E}_o = \frac{2(2M + r_o(cr_o - 1))^2}{r_o(6M + r_o(cr_o - 2))}. \quad (10)$$

For the degenerate case which is defined by $c = \frac{1}{8M}$ or $r_+ = r_-$. The energy and azimuthal angular momentum corresponding to r_o are

$$L_{zo} = \frac{\sqrt{\frac{r_o^2}{8M} - 2M}}{\sqrt{\frac{1}{8M} + \frac{6M - 2r_o}{r_o^2}}}, \quad (11)$$

and

$$\mathcal{E}_o = \frac{2(2M + r_o(\frac{r_o}{8M} - 1))^2}{r_o(6M + r_o(\frac{r_o}{8M} - 2))}. \quad (12)$$

The ISCO is defined by $r_o = 4M$ which is the convolution point of the effective potential [33]. We have not restricted ourself to this local minima at r_o because it depends on the applied condition which we will discuss later in section VIII.

Now consider the particle is in a ISCO and collides with another particle, the later one is coming from the rest position at infinity as a freely falling particle. After collision between particles, three cases are possible for the particles: (i) remain bounded around black hole, (ii) capture by black hole and (iii) escape to infinity. The results depend on the collision process. For small changes in energy and angular momentum, orbit of the particle

is slightly perturbed but particle remains bounded. For larger change in energy and angular momentum, it can go away from initial path and could be captured by black hole or escape to infinity.

After the collision particle should have new values of energy and azimuthal angular momentum and the total angular momentum. We simplify the problem by applying the following conditions: (i) the azimuthal angular momentum does not change and (ii) initial radial velocity remains same after collision. Under these conditions only energy can change by which we can determine the motion of the particle. After collision particle acquires an escape velocity (v_{\perp}) in orthogonal direction of the equatorial plane [12].

After collision the total angular momentum and energy of the particle become (at $\theta = \frac{\pi}{2}$)

$$L^2 = r_o^2 v_{\perp}^2 + L_z^2, \quad (13)$$

$$\mathcal{E} = \left[f(r) \left(1 + \frac{(L_z + r v_{\perp})^2}{r^2} \right) \right]^{\frac{1}{2}}. \quad (14)$$

These new values of angular momentum and energy are greater from their values before collision because during collision colliding particle may impart some of its energy to the orbiting particle. We get the expression (15) for velocity v from Eq. (14) after solve it for v to get

$$v_{\perp}^{\text{esc}} \geq \frac{L_z r (r - 2M - cr^2) + \sqrt{r^4 (r(1 - cr) - 2M)(2M + r(cr + \mathcal{E}^2 - 1))}}{r^2 (2M + r(cr - 1))}, \quad (15)$$

particle would escape if $|v_{\perp}^{\text{esc}}| \geq v_{\perp}$.

III. DYNAMICS OF A CHARGED PARTICLE

We investigate how does the motion of a charged particle is effected by both magnetic field in the black hole exterior and gravitational field. The general Killing vector equation is [34]

$$\square \xi^{\mu} = 0, \quad (16)$$

where ξ^{μ} is a Killing vector. Eq. (16) coincides with the Maxwell equation for 4-potential A^{μ} in the Lorentz gauge $A^{\mu}{}_{;\mu} = 0$. The special choice [35]

$$A^{\mu} = \frac{\mathcal{B}}{2} \xi_{(\phi)}^{\mu}, \quad (17)$$

corresponds to the test magnetic field, where \mathcal{B} is the magnetic field strength. The 4-potential is invariant under the symmetries which corresponds to the Killing vectors as discussed above, i. e.,

$$L_\xi A_\mu = A_{\mu,\nu} \xi^\nu + A_\nu \xi^\nu_{;\mu} = 0. \quad (18)$$

A magnetic field vector is defined as [32]

$$\mathcal{B}^\mu = -\frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} u_\nu, \quad (19)$$

where

$$\epsilon^{\mu\nu\lambda\sigma} = \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}, \quad \epsilon_{0123} = 1, \quad g = \det(g_{\mu\nu}). \quad (20)$$

$\epsilon^{\mu\nu\lambda\sigma}$ is the Levi Civita symbol. The Maxwell tensor is defined as

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu;\mu} - A_{\mu;\nu}. \quad (21)$$

For a local observer at rest in the space-time (1), the non-vanishing components of 4-velocity are

$$u_0^\mu = \frac{1}{\sqrt{f(r)}} \xi_{(t)}^\mu, \quad u_3^\mu = \frac{1}{\sqrt{r^2 \sin^2 \theta}} \xi_{(\phi)}^\mu. \quad (22)$$

The other two components u_1^μ and u_2^μ are zero at the turning point ($\dot{r} = 0$). From equations (19) – (22) we have obtained the magnetic field given below

$$\mathcal{B}^\mu = \mathcal{B} \frac{1}{\sqrt{f(r)}} \left[\cos \theta \delta_r^\mu - \frac{\sin \theta \delta_\theta^\mu}{r} \right]. \quad (23)$$

Here we considered magnetic field to be directed along the vertical (z -axis) and $\mathcal{B} > 0$.

The Lagrangian of the particle of mass m and electric charge q moving in an external magnetic field of a curved space-time is given by [36]

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} u^\mu u^\nu + \frac{q A_\mu}{m} u^\mu, \quad (24)$$

and generalized 4-momentum of the particle $p_\mu = m u_\mu + q A_\mu$. The new constants of motion are

$$\dot{t} = \frac{\mathcal{E}}{f(r)}, \quad \dot{\phi} = \frac{L_z}{r^2 \sin^2 \theta} - B, \quad (25)$$

here

$$B \equiv \frac{q \mathcal{B}}{2m}. \quad (26)$$

By using these constants of motion in the Lagrangian we get the dynamical equations for θ and r respectively

$$\ddot{\theta} = B^2 \sin \theta \cos \theta - \frac{2}{r} \dot{r} \dot{\theta} - \frac{L_z^2 \cos^2 \theta}{r^4 \sin^3 \theta}, \quad (27)$$

$$\begin{aligned} \ddot{r} = & -\frac{(2M + r(cr - 1))^2}{(-2M + r - cr^2)} \left(\frac{L_z^2}{r^4 \sin^2 \theta} \right) + \frac{(2M + cr^2)}{(2M + cr^2 - r)^2} (\mathcal{E} + \dot{r}^2) \\ & + \frac{(2M + r(cr - 1))^2}{(-2M + r - cr^2)} (B^2 \sin^2 \theta + \dot{\theta}^2). \end{aligned} \quad (28)$$

By using normalization condition we get

$$\mathcal{E}^2 = \dot{r}^2 + r^2 f(r) \dot{\theta}^2 + f(r) \left[1 + r^2 \sin^2 \theta \left(\frac{L_z}{r^2 \sin^2 \theta} - B \right)^2 \right]. \quad (29)$$

From Eq. (29) we can write the effective potential as

$$U_{eff} = f(r) \left[1 + r^2 \sin^2 \theta \left(\frac{L_z}{r^2 \sin^2 \theta} - B \right)^2 \right]. \quad (30)$$

The above equation is a constraint i.e. if it is satisfied initially, then it is always valid, provided that $\theta(\tau)$ and $r(\tau)$ are controlled by equation (27) and (28).

Let us discuss the symmetries of Eqs. (24) – (29), these equation are invariant under the transformation given below

$$\phi \rightarrow -\phi, \quad L_z \rightarrow -L_z, \quad B \rightarrow -B. \quad (31)$$

Therefore, without losing the generality, we consider the positively charged particle. The trajectory of a negatively charged particle is related to positive charge's trajectory by transformation (31). If we make a choice $\mathcal{B} > 0$ then we will have to study both cases when $L_z > 0$, $L_z < 0$. They are physically different: the change of sign of L_z means the change of direction of the Lorentz force on the particle.

System of Eqs. (24) – (29) is invariant with respect to reflection ($\theta \rightarrow \pi - \theta$). This transformation retains the initial position of the particle and changes $v_\perp \rightarrow -v_\perp$. Therefore, it is sufficient to consider only the positive value of v_\perp .

IV. DIMENSIONLESS FORM OF THE DYNAMICAL EQUATIONS

Before integrating our dynamical equations of r and θ numerically we make these equations dimensionless by introducing following dimensionless quantities [32, 37]:

$$2m = r_d, \quad \sigma = \frac{\tau}{r_d}, \quad \rho = \frac{r}{r_d}, \quad \ell = \frac{L_z}{r_d}, \quad b = Br_d, \quad c_1 = cr_d. \quad (32)$$

Eqs. (27) and (28) acquire the form

$$\frac{d^2\theta}{d\sigma^2} = b^2 \sin\theta \cos\theta - \frac{2}{\rho} \frac{d\rho}{d\sigma} \frac{d\theta}{d\sigma} + \frac{\ell^2 \cos^2\theta}{\rho^4 \sin^3\theta}, \quad (33)$$

$$\begin{aligned} \frac{d^2\rho}{d\sigma^2} = & (1 + c_1\rho^2 - \rho) \frac{\ell^2}{\rho^4 \sin^2\theta} + \frac{(1 + c_1\rho^2)}{(1 + c_1\rho^2 - \rho)^2} \left(\mathcal{E} + \left(\frac{d\rho}{d\sigma} \right)^2 \right) \\ & - (1 + c_1\rho^2 - \rho) \left(b^2 \sin^2\theta + \left(\frac{d\theta}{d\sigma} \right)^2 \right). \end{aligned} \quad (34)$$

These are very complicated equations and to study such problems 3D numerical simulation of the magnetohydrodynamics (MHD) in a strong gravitational field is required [38]. For simplicity we fix $\theta = \frac{\pi}{2}$ then the Eq. (33) is satisfied and the Eq. (34) becomes

$$\frac{d^2\rho}{d\sigma^2} = (1 + c_1\rho^2 - \rho) \left(\frac{\ell^2}{\rho^4} - b^2 \right) + \frac{(1 + c_1\rho^2)}{(1 + c_1\rho^2 - \rho)^2} \left(\mathcal{E} + \left(\frac{d\rho}{d\sigma} \right)^2 \right). \quad (35)$$

We have solved Eq. (35) numerically by using the built in command NDSolve in Mathematica 8.0. We have obtained the interpolating function as a solution of Eq. (35) and plotted the derivative of interpolating function (radial velocity of the particle) as a function of σ in Fig. 1, it shows that $\rho'(\sigma)$ is increasing as we increase the dimensionless parameter σ .

Eqs. (29) and (30) become

$$\mathcal{E}^2 = \left(\frac{d\rho}{d\sigma} \right)^2 + \rho^2 \left(1 - c_1\rho - \frac{1}{\rho} \right) \left(\frac{d\theta}{d\sigma} \right)^2 + U_{\text{eff}}. \quad (36)$$

$$U_{\text{eff}} = \left(1 - c_1\rho - \frac{1}{\rho} \right) \left[1 + \rho^2 \left(\frac{\ell}{\rho^2 \sin^2\theta} - b \right)^2 \right]. \quad (37)$$

The energy of the particle moving around the black hole in an orbit of radius ρ_o at the equatorial plane is given by

$$U_{\text{eff}} = \mathcal{E}_o^2 = \left(1 - c_1\rho - \frac{1}{\rho} \right) \left[1 + \frac{(\ell - b\rho_o^2)^2}{\rho_o^2} \right]. \quad (38)$$

Solving $\frac{dU_{\text{eff}}}{d\rho} = 0$ and $\frac{d^2U_{\text{eff}}}{d\rho^2} = 0$ simultaneously, we calculate b and ℓ in term of ρ ,

$$\frac{dU_{\text{eff}}}{d\rho} = \rho^2 - 2b\ell\rho^2 - b^2\rho^4 - 2b^2\rho^5(c_1 - 1) + \ell^2(3 + 2\rho(c_1 - 1)), \quad (39)$$

and

$$\frac{d^2U_{\text{eff}}}{d\rho^2} = 2[2b\ell\rho^2 - \rho^2 + b^2\rho^5(1 - c_1) - 3\ell^2(2 + \rho(c_1 - 1))]. \quad (40)$$

The obtained expressions for b and ℓ are

$$b = \frac{1}{2\sqrt{2\rho^4(1+\rho(c_1\rho-1))^2[3+\rho(\rho(4+c_1(18+\rho(3c_1\rho-18)))-10)]}} \times \left[\sqrt{\rho^4[1-\rho(3+c_1\rho(\rho(3c_1\rho-1)-6))](\rho(1+c_1\rho(6+\rho(c_1\rho-3)))-3)^3} + \rho^2(\rho(1+c_1\rho(6+\rho(c_1\rho-3)))-3)(3+\rho(\rho(4+c_1(14+3\rho(c_1\rho-3)))-9)) \right]^{\frac{1}{2}} \quad (41)$$

and

$$\ell = \frac{\sqrt{\rho^4[1-\rho(3+c_1\rho(\rho(3c_1\rho-1)-6))](\rho(1+c_1\rho(6+\rho(c_1\rho-3)))-3)^3}}{(\rho(1+c_1\rho(6+\rho(c_1\rho-3)))-3)^2} \times \left[\frac{1}{2\sqrt{2\rho^4(1+\rho(c_1\rho-1))^2[3+\rho(\rho(4+c_1(18+\rho(3c_1\rho-18)))-10)]}} \times \left\{ \sqrt{\rho^4[1-\rho(3+c_1\rho(\rho(3c_1\rho-1)-6))](\rho(1+c_1\rho(6+\rho(c_1\rho-3)))-3)^3} + \rho^2(\rho(1+c_1\rho(6+\rho(c_1\rho-3)))-3)(3+\rho(\rho(4+c_1(14+3\rho(c_1\rho-3)))-9)) \right\}^{\frac{1}{2}} \right] \quad (42)$$

In Fig. 2 we have plotted magnetic field b against ρ for different values of c_1 . It can be seen

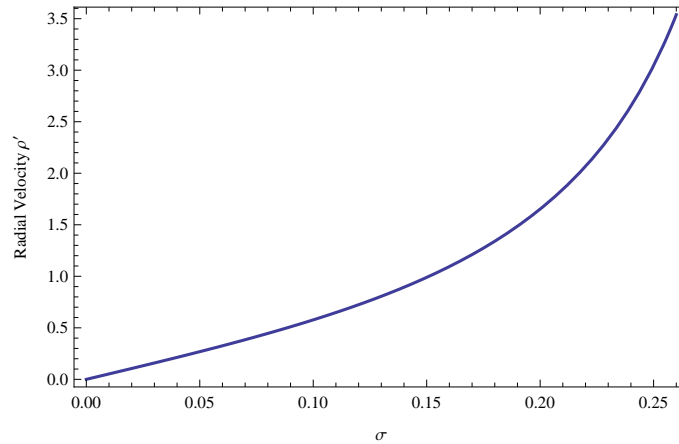


FIG. 1: Radial velocity as a function of σ for $\ell = 5$, $\mathcal{E} = 1$, $b = 0.25$ and $c_1 = 0.125$.

that the strength of magnetic field is increasing for large value of c_1 . We can conclude that the presence of dark energy strengthens the magnetic field present in the vicinity of black hole. The strength of magnetic field is decreasing away from the black hole. In Fig. 34 we have plotted the ℓ for different value of c_1 . It can be seen that larger the value of c_1 greater will be ℓ . So, presence of dark energy increases the Lorentz force on the orbiting particle.

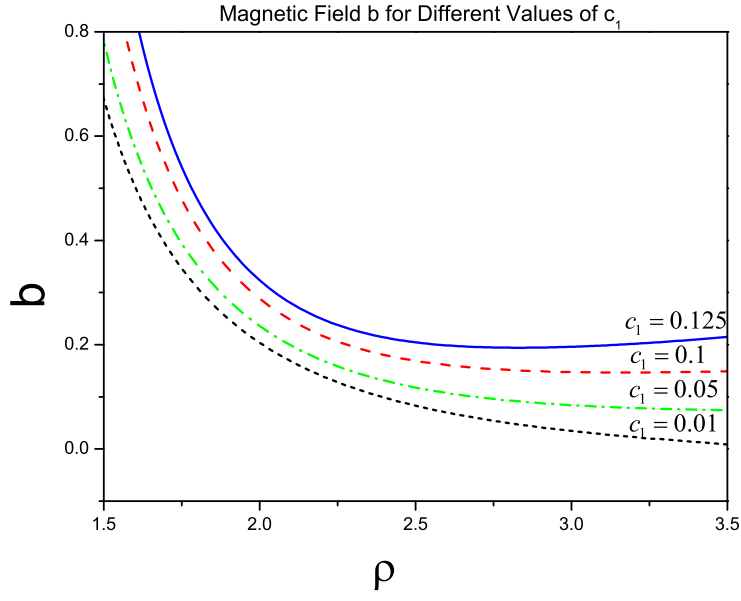


FIG. 2: Magnetic field b as a function of ρ for different value of c_1 .

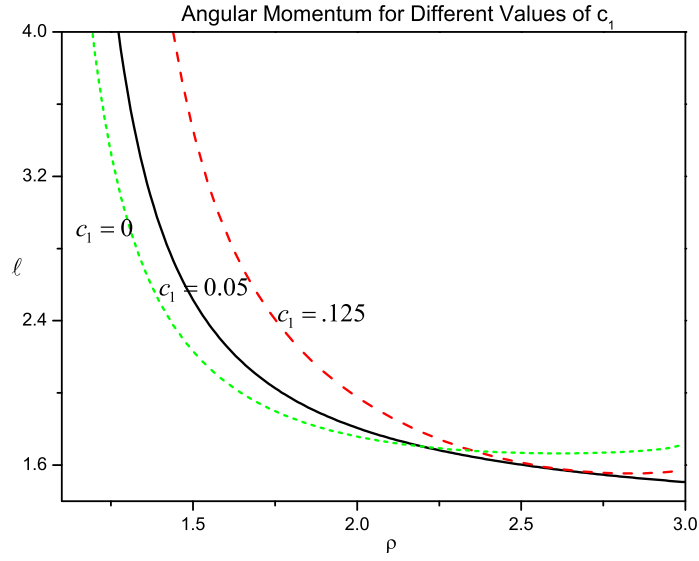


FIG. 3: Figure shows the behavior of angular momentum as a function of ρ for different value of c_1 .

The angular momentum ℓ , for ISCO, as function of magnetic field b is shown in Fig. 4 and Fig. 5. Lorentz force is attractive if $\ell > 0$, corresponds to Fig. 4 and it is repulsive if $\ell < 0$, corresponds to Fig. 5.

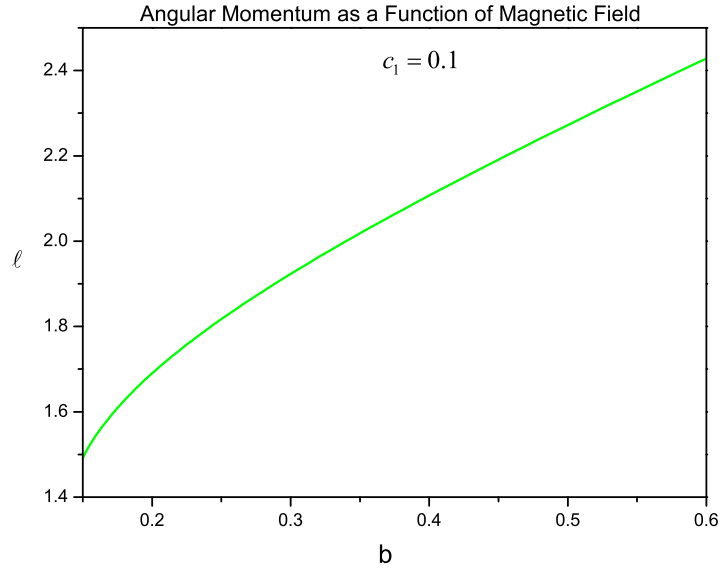


FIG. 4: Behavior of angular momentum ℓ_+ vs magnetic field b .

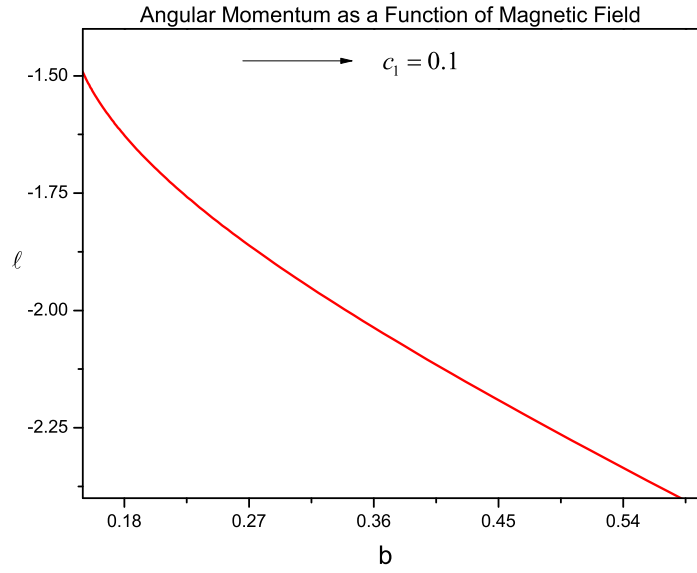


FIG. 5: Angular momentum ℓ_- against magnetic field b .

As we did before in the case of a neutral particle, we assume that the collision does not change the azimuthal angular momentum of the particle but it does change the velocity

$v_{\perp} > 0$. Due to this, the particle energy changes, $\mathcal{E}_o \rightarrow \mathcal{E}$, and is given by

$$\mathcal{E} = \sqrt{\left(1 - c_1\rho - \frac{1}{\rho}\right) \left[1 + \rho^2 \left(\frac{\ell + \rho v_{\perp}}{\rho^2} - b\right)^2\right]}. \quad (43)$$

Escape velocity of the particle obtained from Eq. (43) is given below

$$v_{\perp}^{\text{esc}} \geq \frac{1}{\rho^2(1 + \rho(c_1\rho - 1))} [\rho(1 + \rho(c_1\rho - 1))(b\rho^2 - \ell) \pm \sqrt{\rho^4(\rho(1 - c_1\rho) - 1)(1 + \rho(\mathcal{E}^2 + c_1\rho - 1))}]. \quad (44)$$

Behavior of escape velocity is discussed graphically in section VIII.

V. CENTER OF MASS ENERGY OF THE COLLIDING PARTICLES

A. In the absence of magnetic field ($B=0$)

First we consider that the two neutral particles of masses m_1 and m_2 coming from infinity collide near the black hole when there is no magnetic field. The collision energy of the particles of masses $m_1 = m_2 = m_o$ in the center of mass frame is defined as [15]

$$E_{cm} = m_o \sqrt{2} \sqrt{1 - g_{\mu\nu} u_1^{\mu} u_2^{\nu}}, \quad (45)$$

where

$$u_i^{\mu} \equiv \frac{dx^{\mu}}{d\tau}, \quad i = 1, 2 \quad (46)$$

is the 4-velocity of each of the particles. Using Eqs. (4), (5), and (7), in Eq. (45) we get the CME for the neutral particle, falling freely from rest at infinity, given below

$$E_{cm} = m_o \sqrt{2} \left[2 + \left(\frac{L_1^2 + L_2^2}{2r^2}\right) \left(\frac{\mathcal{E}^2 + f(r)}{\mathcal{E}^2}\right) + (L_1 L_2) \left(\frac{f(r)(L_1 L_2) - 2r^2 \mathcal{E}^2}{2\mathcal{E}^2 r^4}\right) + \frac{f(r)}{2\mathcal{E}^2} \right]^{1/2} \quad (47)$$

We are interested to find out the CME of the particles near the horizon, so taking $f(r) = 0$ we get

$$E_{cm} = 2m_o \sqrt{1 + \frac{1}{4r_h^2} (L_1 - L_2)^2}, \quad (48)$$

where $r_h = \frac{1 \pm \sqrt{1 - 8Mc}}{2c}$ represents the horizons of the black hole, obtained earlier. The expression of CME obtained in Eq. (48) could be infinite if the angular momentum of one of the particles gets infinite value, but it would not allow the particle to reach the horizon of the black hole. Thus the CME in Eq. (48) can not be unlimited.

B. In the presence of magnetic field

For a charged particle moving around the black hole we have obtained the constants of motion defined in Eq. (25), using it with normalization condition of the metric we get equation of motion of the particle

$$\dot{r}^2 = \mathcal{E}^2 - f(r)[1 + r^2(\frac{L}{r^2} - B)^2]. \quad (49)$$

Using Eqs. (25), (49) with Eq. (45) we get the expression for CME of the charged particles coming from infinity, colliding near the black hole

$$\begin{aligned} E_{cm} = & m_o \sqrt{2} \left[2 + (L_1^2 + L_2^2) \left[\frac{f(r) + \mathcal{E}^2}{2r^2 \mathcal{E}^2} + \frac{f(r)B^2}{2\mathcal{E}^2} \right] - (L_1 + L_2) \left[B \left(\frac{\mathcal{E}^2 + f(r)}{\mathcal{E}^2} \right) + \frac{f(r)r^2 B^3}{\mathcal{E}^2} \right] + \right. \\ & L_1 L_2 \left[\frac{f(r)}{2\mathcal{E}^2 r^4} \left(L_1 L_2 + 2Br^2(L_1 + L_2) + 4B^2 r^4 \right) - \frac{1}{r^2} \right] + \\ & \left. B^2 r^2 \left(\frac{\mathcal{E}^2 + f(r)}{\mathcal{E}^2} \right) + \frac{f(r)}{2\mathcal{E}^2} + \frac{f(r)r^4 B^4}{2\mathcal{E}^2} \right]^{1/2}, \end{aligned} \quad (50)$$

near horizon i.e. at $f(r) = 0$, Eq. (50) becomes

$$E_{cm} = 2m_o \left[1 + \frac{1}{4r_h^2} (L_1 - L_2)^2 - \frac{B}{2} [(L_1 + L_2) - Br^2] \right]^{1/2}, \quad (51)$$

where r_h represent the horizons of the black hole. The CME in Eq. (51) could be infinite, if the angular momentum of one of the particles has infinite value, for which the particle can not reach the horizon of the black hole. Thus the CME defined in Eq. (51) is some finite energy.

VI. LYAPUNOV EXPONENT FOR THE INSTABILITY OF ORBIT

We can check the instability of circular orbit by Lyapunov exponent, λ , which is given by [31]

$$\lambda = \sqrt{\frac{-U''_{eff}(r_o)}{2\dot{t}(r_o)^2}} \quad (52)$$

$$\lambda = \sqrt{\frac{(2M + r(-1 + cr))(-2Mr^2 + 4BLMr^2 + B^2 r^5(1 - 3cr) - L^2(12M + r(cr - 3)))}{L^2 r^4}} \quad (53)$$

Behavior of λ as a function of c is shown in Fig. 6 and Fig. 7. It can be seen from these figures that instability of circular orbits is less for non zero c compared to the case when

$c = 0$, i.e. around Schwarzschild black hole. In Fig. 8, λ is compared for three different types of black hole, Schwarzschild black hole, Schwarzschild black hole immersed in magnetic field and Kiselev black hole immersed in magnetic field. It is observed that with non zero c or B , stability is more as compared to Schwarzschild black hole, λ is smaller for Kiselev black hole as compared to schwarzschild black hole and it gets more small for Kiselev black hole surrounded by both dark energy and magnetic field. In all the figures here after, keep in mind that $b = Br_d$.

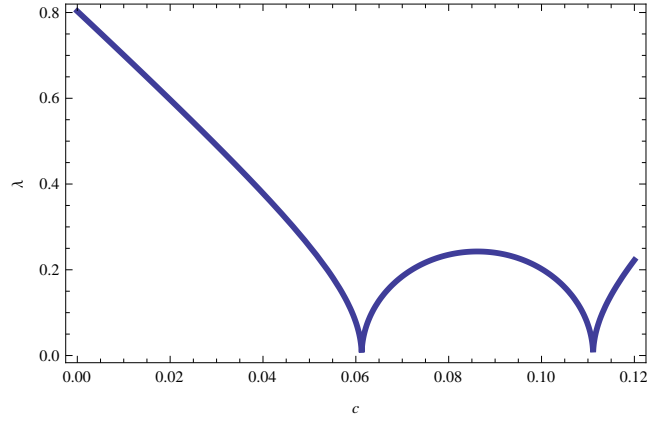


FIG. 6: Lyapunov exponent as a function of c for massive particle. Here $r = 3$, $M = 1$, $L = 3.22$, and $b = 0.25$

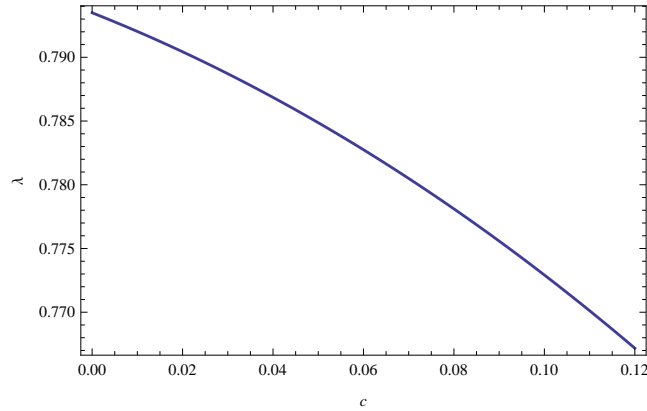


FIG. 7: Lyapunov exponent as a function of c for massless particles. Here $r = 3$, $M = 1$, $L = 3.22$, and $b = 0.25$

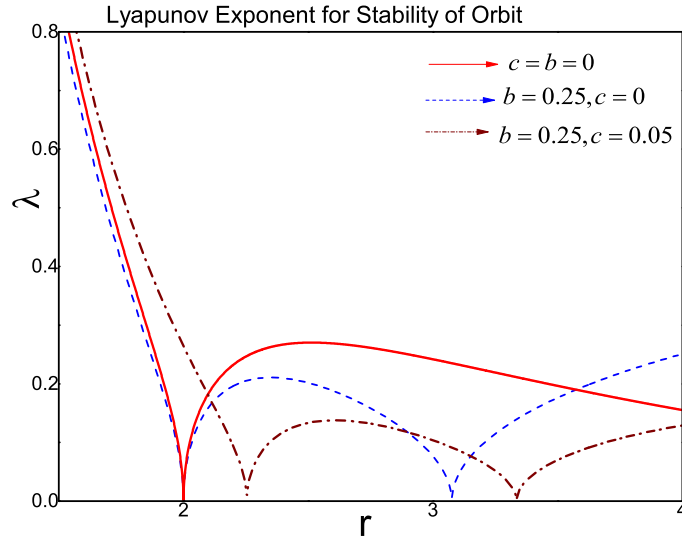


FIG. 8: Lyapunov exponent for different values of c and b as a function of radial coordinate r for $M = 1$, and $L = 3.22$.

VII. EFFECTIVE FORCE ON THE PARTICLE

Effective force on a particle, computed with the help of effective potential is given by

$$F = \frac{-1}{2} \frac{dU_{\text{eff}}}{dr}, \quad (54)$$

$$F = -\frac{M}{2r^4} (6L^2 - 4BLr^2 - 2B^2r^4) + \frac{1}{2r^4} (2L^2r - 2r^2 - 2Br^5) - \frac{c}{2r^4} (r^2L^2 + 2BLr^4 - r^4 - 3B^2r^6). \quad (55)$$

First and third terms of equation (55) are responsible for attractive force if $(6L^2 > -4BLr^2 - 2B^2r^4)$ and $(r^2L^2 + 2BLr^4 > -r^4 - 3B^2r^6)$ respectively. Here third term is appearing due to quintessence matter, hence the quintessence is behaving as a source of attraction inside the horizons. Second term is repulsive if $(2L^2r > -2r^2 - 2Br^5)$. In case of massless particles the first term and the dark energy term are purely responsible for attractive force contribution (without any condition) and remaining term is responsible for repulsive force contribution [28].

In Fig. 9 we have plotted the effective force for an orbiting particle in a circular orbit. In Fig. 9 F_{max} correspond to unstable circular orbits and F_{min} corresponds to stable circular orbits. From this figure we can deduce that there is no stable circular orbits for photon, it

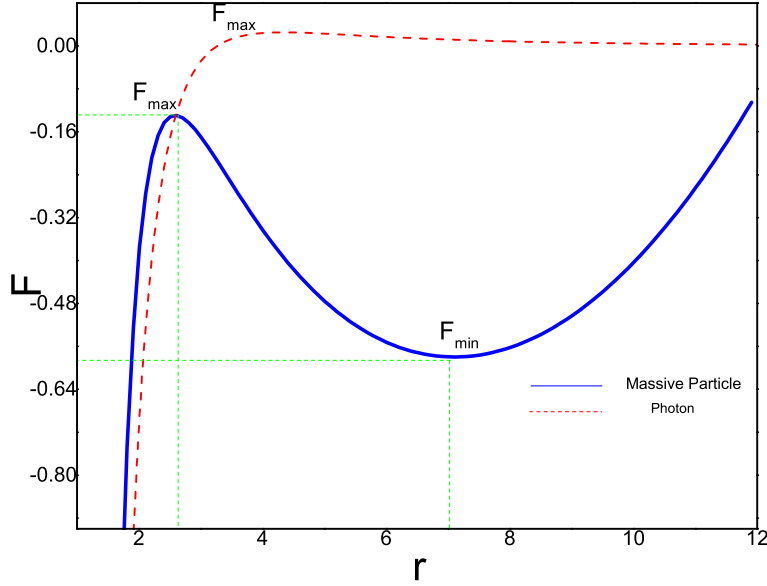


FIG. 9: Effective force as a function of r . Here $c = 0.05$, $M = 1$, $L = 3.22$ and $b = 0.5$.

only exist for massive particle. For the rotational (angular) variable

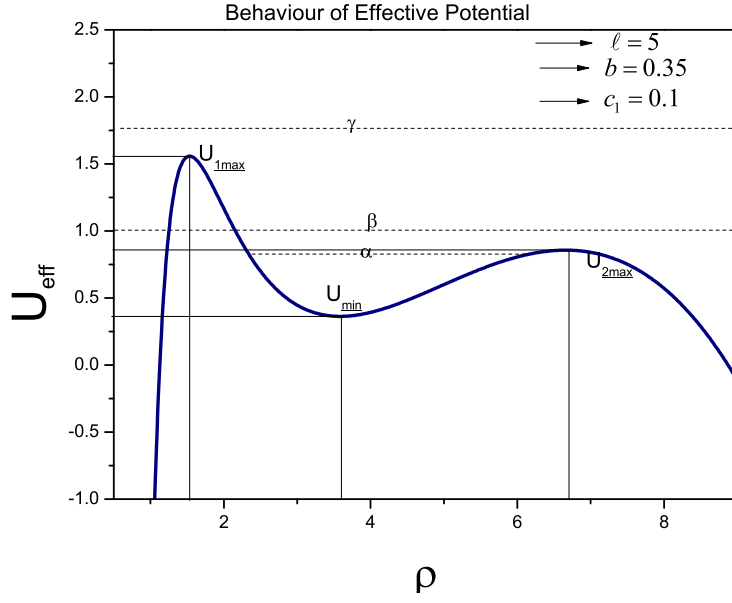
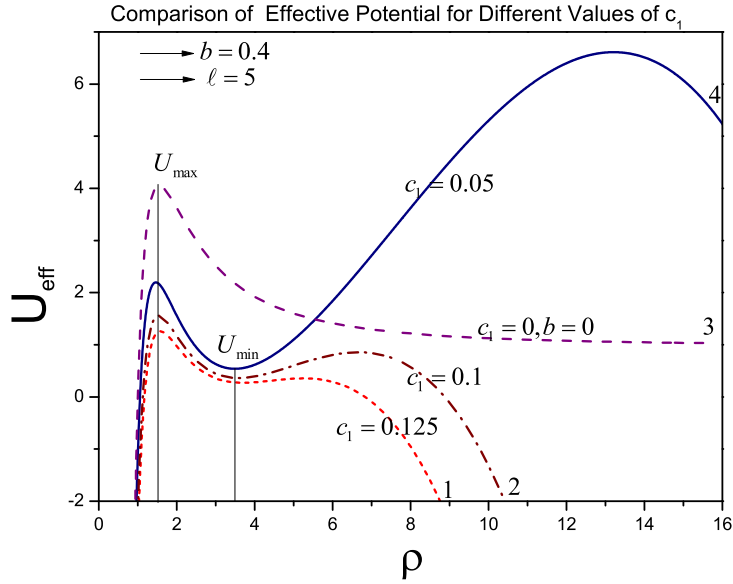
$$\frac{d\phi}{d\tau} = \frac{L_z}{r^2} - b. \quad (56)$$

If right hand side of equation (56) is positive ($L_z > b$), then the Lorentz force on the particle is repulsive (particle moving in anticlockwise direction), and if right hand side is negative ($L_z < b$), then the Lorentz force is attractive (clockwise rotation).

VIII. TRAJECTORIES FOR EFFECTIVE POTENTIAL AND ESCAPE VELOCITY

Behavior of effective potential is demonstrated by plotting it vs ρ , in Fig. 10. The horizontal line α with $\mathcal{E} < 1$ corresponds to bound motion, this is the analogue of elliptical motion in Newtonian theory. The trajectories of the particle is not closed in general. The line segment β with $\mathcal{E} > 1$ corresponds to a particle coming from infinity and then move back to infinity (hyperbolic motion).

The line γ does not intersects with the curve of effective potential and passes above its maximum value U_{1max} . It corresponds to particle which is falling into the black hole

FIG. 10: The effective potential versus ρ .FIG. 11: The effective potential as a function ρ , for different value of c .

(captured by the black hole) and $U_{1\text{max}}$ and $U_{2\text{max}}$ correspond to unstable orbits and U_{min} refers to a stable circular orbit.

In Fig. 11 we have compared the effective potentials for different values of c . One can notice as the value of c increases the maxima and minima of effective potential shifted

downward. Here U_{max} and U_{min} corresponds to unstable and stable orbits of the particle around the black hole respectively. In Fig. 11, curve 3 represents the Schwarzschild effective potential [39]. Therefore, one can say that the dark energy acts to decrease the effective potential. We can conclude that force on the particle due to dark energy is attractive. Hence the possibility for a particle to be captured by the black hole is greater due to presence of dark energy as compare to the case when $c = 0$. Effective potential vs ρ is plotted in Fig. 12 for different values of magnetic field b . One can notice from the Fig. 12 that orbits are more stable in the presence of magnetic field as compare to the case when magnetic field is absent, $b = 0$. It can also be seen that the local minima of the effective potential which corresponds to ISCO is shifting toward the horizon which is in agreement with [35, 40]. We have compared the effective potential for massive and massless particles (photons) in Fig. 13. For photon there is no stable orbit as there is no minima for $\ell = 0$ represented by plot 3 in Fig. 13. While for the massive particle there are local minima U_{min1} and U_{min2} which correspond to stable orbits. It can also be concluded that the particle having larger value of angular momentum ℓ can escape easily as compare to the particle with lesser value of angular momentum ℓ .

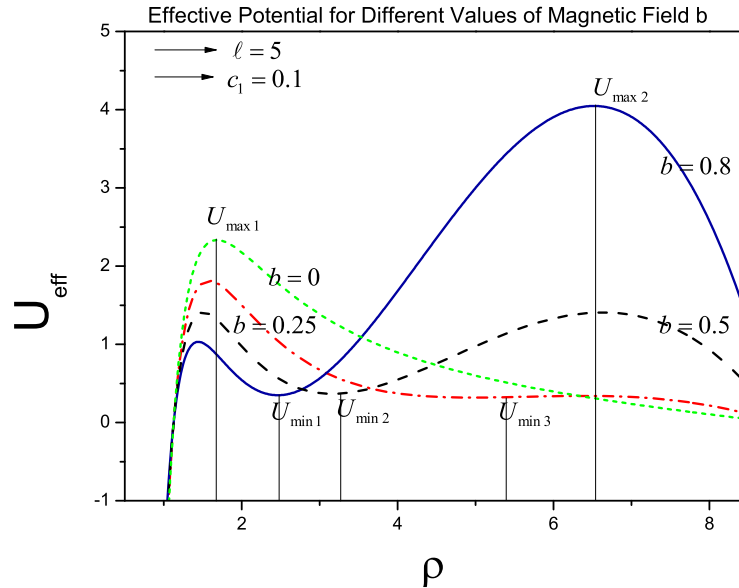


FIG. 12: The effective potential against ρ , for different value of magnetic field b .

Fig. 14 explains the behavior of escape velocity of the particle moving around the black hole. The shaded region in Fig. 14 corresponds to escape velocity of the particle and the

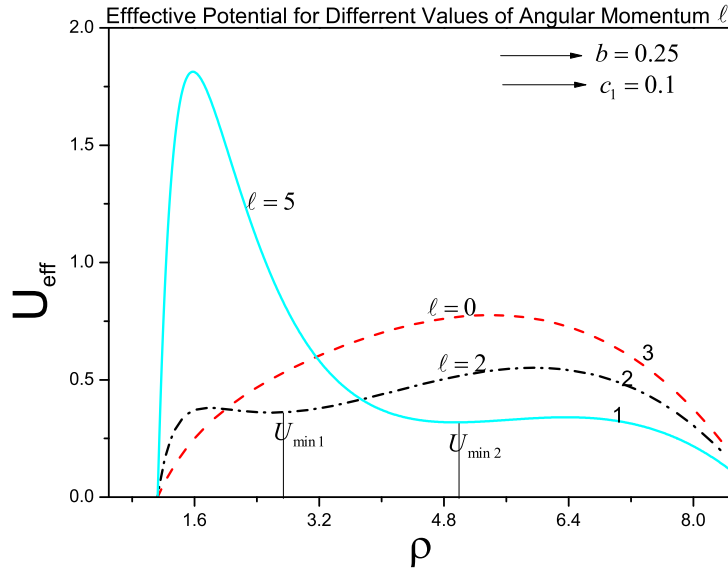


FIG. 13: The effective potential against ρ , for different value of angular momentum ℓ .

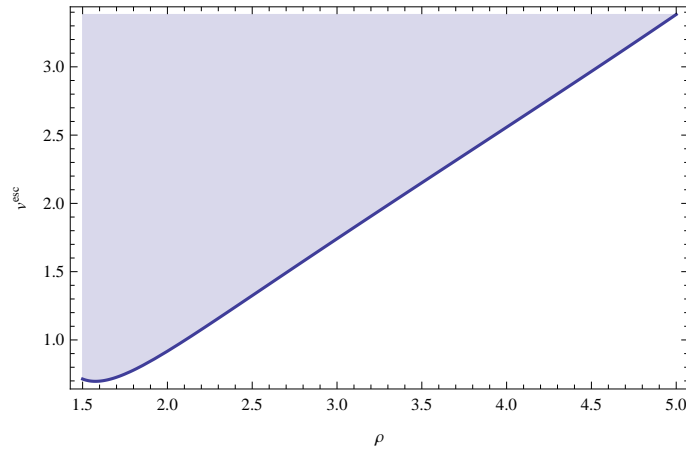


FIG. 14: The escape velocity as a function of ρ for $\ell = 3.22$, $b = 0.50$ and $c_1 = 0.10$.

solid curved line represents the minimum velocity required to escape from the vicinity of the black hole and the unshaded region is for bound motion around the black hole. In Fig. 15 we have plotted the escape velocity for different values of energy \mathcal{E} . Fig. 15 shows that the possibility for the particle having greater energy has more possibility to escape from the vicinity of black hole as compared to the particle with lesser value of energy. Escape velocity for different values of c_1 is plotted in Fig. 16, it shows that greater the value of c greater will be the escape velocity of the particle, provided that the energy of the orbiting

particle after collision is less than U_{max} otherwise it would be captured by the black hole. One can conclude that the presence of dark energy might play a crucial role in the transfer mechanism of energy to the particle during its motion in the ISCO. In Fig. 17 a comparison of the escape velocities for different values of magnetic field b , is done. It can be seen that greater the strength of magnetic field the possibility for a particle to escape increases. It can be concluded that the key role in the transfer mechanism of energy to the particle for escape from the vicinity of black hole is played by the magnetic field which is present in the accretion disc. This is in agreement with the result of [13, 14].

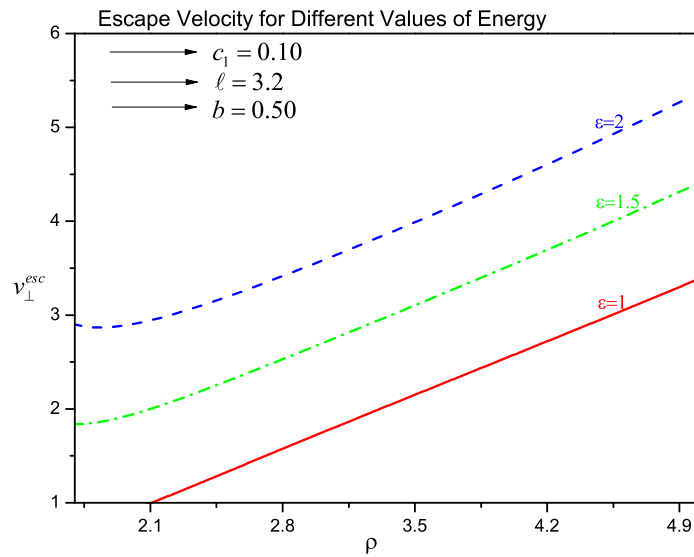


FIG. 15: Plot of escape velocity v_{esc} as a function of ρ for different values of energy \mathcal{E} .

IX. SUMMARY AND CONCLUSION

- We have studied the dynamics of a neutral and a charged particle in the vicinity of Schwarzschild black hole surrounded by quintessence matter. It is known that the quintessence is the candidate for dark energy and the black hole metric which we have studied was derived by Kiselev [27].
- We have studied the motion of a neutral particle in the absence of magnetic field and the dynamics of a charged particle in the presence of magnetic field in the vicinity of black hole in detail.

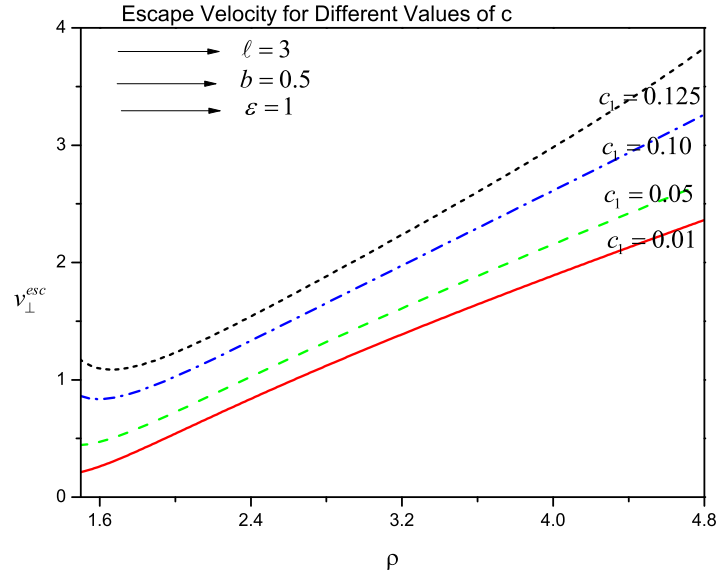


FIG. 16: Plot of escape velocity v_{esc} against ρ for different values of c_1 .

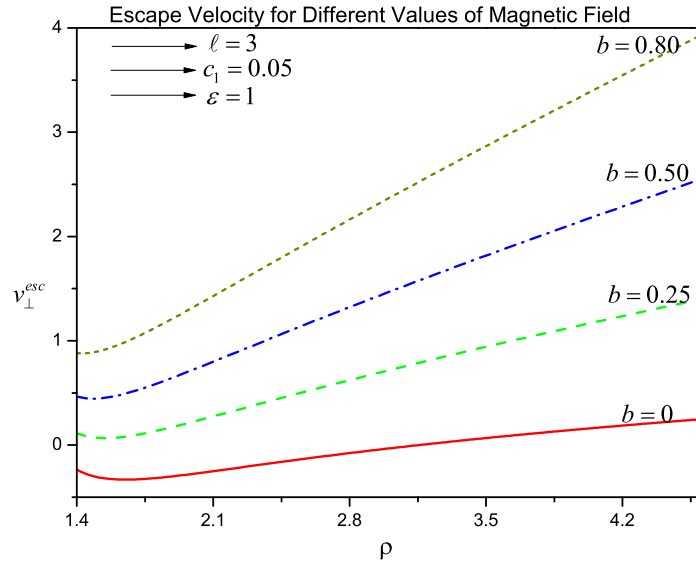


FIG. 17: Plot of escape velocity v_{esc} vs ρ for different values of magnetic field b .

- We have discussed the energy conditions for the stable circular orbits and for unstable circular orbits around the black hole.
- The ISCO for a massive neutral particle around a Schwarzschild-like black hole occurs at $r = 4M$.

- We have found that ISCO shifts closer to the event horizon due to presence of dark energy and magnetic field as compared to Schwarzschild black hole. This is the indication that the force due to dark energy is attractive which is in agreement with the results of [28].
- Center of mass energy expressions are derived for the colliding particles near horizons. It is found that CME is finite for the particle colliding in the vicinity of Kiselev solution.
- We have derived the formula for escape velocity of the particles, after collision.
- The equations of motion have been solved numerically and we have plotted the radial velocity of the particle.
- The Lyapunov exponent, which gives the instability time scale for the geodesics of the particle, is calculated. Therefore we have concluded that the instability of the circular orbits around Schwarzschild black hole is more compared to Kiselev solution, in the presence of magnetic field.
- We have derived the effective force acting on the particle due to dark energy and mentioned the conditions under which the force is attractive or repulsive.

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- [1] S. Perlmutter et. al., *Astrophys. J.* **517** 565 (1999).
 - [2] D. N. Spergel, *Astrophys. J. Suppl.* **170** 377 (2007).
 - [3] M. Tegmark, *Phys. Rev. D* **69** 103501 (2004).
 - [4] U. Seljak, *Phys. Rev. D* **71** 103515 (2005).
 - [5] S. M. Carroll, *Phys. Rev. Lett.* **81** 3067 (1998).
 - [6] M. Jamil, A. Qadir, *Gen. Rel. Grav.* **43**, 1069 (2011); B. Nayak, M. Jamil, *Phys. Lett. B* **709**, 118 (2012); M. Jamil, D. Momeni, K. Bamba, R. Myrzakulov, *Int J. Mod. Phys D* **21**, 1250065 (2012); M. Jamil, M. Akbar, *Gen. Rel. Grav.* **43**, 1061 (2011).

- [7] C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. **85** 4438 (2000).
- [8] Hsu, S.D.H.: Phys. Lett. B **594**, 13 (2004); Li, M.: Phys. Lett. B **603**, 1 (2004); A. Pasqua, M. Jamil, R. Myrzakulov, B. Majeed, Phys. Scr. **86**, 045004 (2012); B. Majeed, M. Jamil, A. A. Siddiqui, IJTP, 10.1007/s10773-014-2197-3,(2014).
- [9] R. D Blandford, R. L. Znajek, Mon. Not. Roy. Astron. Soc. **179**, 433 (1977).
- [10] S.Koide, K.Shibata.T.Kudoh, and D.l.Meier, Science **295**, 1688(2002).
- [11] S. Kide, Phys. Rev. **D 67**, 104010 (2003).
- [12] C. Van Borm, M. Spaans, Astron. Astrophys. **553**, L9 (2013).
- [13] J. C. Mckinney, R. Narayan, Mon. Not. Roy. Astron. Soc. **375**, 523 (2007).
- [14] P. B. Dobbie, Z. Kuncic.G. V .Bicknell, and R. Salmeron, Proceedings of IAU Symposium 259 Galaxies (Tenerife, 2008).
- [15] M. Banados, J. Silk, S. M. West, Phys. Rev. Lett. **103**, 111102 (2009).
- [16] O. B. Zaslavskii, JETP Lett. **92**, 571 (2010).
- [17] S. W. Wei, Y. X. Liu, H. Guo and C. E Fu, Phys. Rev. **D 82**, 103005 (2010).
- [18] O. B. Zaslavskii, Phys. Rev. **D 81**, 044020 (2010).
- [19] Y. Zhu, S. Wu, Y. Jiang, G. Yang, arXiv:1108.1843.
- [20] M. Patil, A. S. Joshi, K. Nakao, M. Kimura, arXiv: 1108.0288.
- [21] S. Gao and C. Zhong, Phys. Rev. **D 84**, 044006 (2011).
- [22] P. J. Mao, R. Li, L. Y. Jia and J. R. Ren, arXiv:1008.2660v3.
- [23] C. Zhong and S. Gao, JETP Lett. **94**, 631 (2011).
- [24] I. Hussain, Mod. Phys. Lett. A, **27**, 1250017 (2012); I. Hussain, Mod. Phys. Lett. A, **27**, 1250068 (2012); I. Hussain, J .Phys. Conf. Scr. **354**, 012007 (2012).
- [25] M. Sharif and N. Haider, Astrophys. Space Sci. **346**, 111 (2013).
- [26] E. J. Copeland, M. Sami and S. Tsujikawa, Int. Jour. Modren. Phys. **D 15** 1753 (2006).
- [27] V. V. Kiselev, Class. Quan. Grav. **20** 1187 (2003).
- [28] S. fernendo, Gen. Relativ. Gravit. **44**, 1857-1879(2012).
- [29] J. Buchner and L. M. Zelenyi, J. Geophys. Res. **94**, 821 (1989).
- [30] J. Buchner and L. M. Zelenyi, J. Geophys. Res. Lett. **17**, 127 (1990).
- [31] V. Cardoso, A. S. Miranda, E. Berti, H. Witech and V. T. Zanchin, Phys. Rev. **D79** 064016 (2009).
- [32] A. M. Al Zahrani, V. P. Frolov, A. A. Shoom, Phys. Rev. **D 87**, 084043 (2013).

- [33] S. Chandrasekher, *The Mathematical Theory of Black Holes* (Oxford University Press, 1983).
- [34] R. M. Wald, *Phys. Rev. D* **10**, 1680 (1974).
- [35] A. N. Aliev and N. Ozdemir, *Mon. Not. Roy. Astron. Soc.* **336**, 241 (1978).
- [36] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975).
- [37] V. P. Frolov and A. A. Shoom, *Phys. Rev. D* **82**, 084034 (2010).
- [38] B. Punsly, *Black Hole Gravitohydrodynamics* (Springer Verlag, Berlin, 2001).
- [39] D. Raine and E. Thomas, *Black Holes An Introduction* (Imperial College Press, 2005).
- [40] G. Prite, *Class. Quantum Grav.* **21**, 3433 (2004).

